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Application of i-Smooth Analysis to Differential Games with Delays ⋆

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Abstract: In this paper we present application of i-smooth analysis to approach-evasion linear differential game with delay. The main goal is to show that according to the methodology of i-smooth analysis one can realize extremal shift procedure by the finite dimensional component of the system state.

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1. INTRODUCTION

Since Isaacs's pioneering work on pursuit-evasion games [Isaacs (1965)], differential games have been extensively studied. To date, a corresponding theory has been developed for various models of differential games. In the present paper we consider *functional-differential games* (FDG) described by differential equations with delay (functional-differential equations). Various aspects of the FDG theory and methods for solution of FDG were considered in the works Friedman (1970); Osipov (1971); Kryazhimskii and Osipov (1973); Osipov and Pimenov (1978); Gomoyunov and Lukoyanov (2012).

Within the theory of positional differential games *an extremal shift method* (an extremal aiming method) was developed by N.N. Krasovski and A.I. Subbotin [Krasovskii and Subbotin (1974)]. The method allows to constructively solve the problem of finding optimal guaranteed control laws in guaranteed control problems, and also proves useful in solving problems of static optimization [Kryazhimskii and Osipov (2004)], trajectory tracking problem [Maksimov (2014)]. The application of the extremal shift method in the construction of control in FDG is difficult to realize, since extremal aiming is carried out on the functional (infinite-dimensional) component. This paper is devoted to an application of i-smooth analysis [Kim (2015)] to realization of extremal shift method for the approach-evasion linear FDG. From the methodology of i-smooth analysis it follows that one can try to build an extremal strategy, aiming only at the finite-dimensional component, and use the functional components in calculating the right-hand sides of the functional-differential equations.

The paper is organized as follows. Sections 2, 3 contain the necessary theoretical background and approach problem statement. In Section 4, the solvability conditions for the approach problem are given.

2. STATEMENT OF THE PROBLEM

We consider the system described by the linear differential equations with delay

$$\dot{x} = A(t)x(t) + A_\tau(t)x(t-\tau) + B(t)u - C(t)v + w(t), \quad (1)$$

$$t_0 \le t \le \vartheta,\ x \in R^n,\ u \in P(t) \subset R^r,\ v \in Q(t) \subset R^s. \quad (2)$$

Here t is the time variable, x is the state vector of system; u and v are control parameters of the first and the second players respectively; $P(t)$ and $Q(t)$ are convex compacts continuous in t on the interval $[t_0, \vartheta]$; $A(t)$, $A_\tau(t)$, $B(t)$, $C(t)$ are $n\times n$, $n\times n$, $r\times n$, $s\times n$ matrices, respectively, with continuous on $[t_0, \vartheta]$ elements; $w(t)$ is n-dimensional piecewise continuous on $[t_0, \vartheta]$ vector-function; $\tau = const > 0$ is the delay value; $H = R^n \times Q[-\tau, 0)$ is the phase space of the system (1).

Position of the system (1) at a time moment t is the pair

$$x_t = \{x(t), x(t+\cdot) = \{x(t+\varsigma), -\tau \le \varsigma < 0\}\},$$

which is the segment $x(\xi), \xi \in [t-\tau, t]$ of the system trajectory in the space H of pairs $h = \{x, y(\cdot)\} \in R^n \times Q[-\tau, 0)$. In H the inner product

$$(h^{(1)}, h^{(2)})_H = (x^{(1)}, x^{(2)})_n + \int_{-\tau}^{0} (y^{(1)}(s), y^{(2)}(s))_n ds$$

and the corresponding norm $\|h\|$ are defined. Here $h^{(1)} = \{x^{(1)}, y^{(1)}(s)\}$, $h^{(2)} = \{x^{(2)}, y^{(2)}(s)\}$, and the symbol $(\cdot,\cdot)_n$ is the inner product on R^n.

It is assumed that sets $M \subset R^n$, $N \subset H$ are defined, a target M set is given. Players have the opposite goals. The approach problem posed for the first player consists of choosing a positional strategy U^0 such that for any motion $x(\cdot) \in X_1(t_0, h_0, U^0)$ the point $x(t)$ should contact

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the set M at some time $T \in [t_0, \vartheta]$ and during the process x_t should remain in the set N. The goal of the second player is to control (using v) the system in a way to avoid approaching to M. Both players form feedback controls basing on information about current position $x_t = \{x(t); x(t+\varsigma), -\tau \le \varsigma < 0\}$ of the system.

3. APPROACH PROBLEM STATEMENT

In this section we consider the game from the position of the first player.

Definition. A program control of the first (second) player is a piece-wise continuous on $[t_\alpha, t_\beta]$ vector function $u(t)$ $(v(t)) : [t_\alpha, t_\beta] \to R^r$, satisfying for all $[t_\alpha, t_\beta]$ the condition $u(t) \in P(t)$ $(v(t) \in Q(t))$.

By $\{u(\cdot)\}$ $(\{v(\cdot)\})$ we denote the sets of all program controls of the first (second) player.

Definition. 1) Strategy U (V) of the first (second) player is a rule which to every position $p = \{t, h\}$ put in line a set $\mathcal{U}(p) = \mathcal{U}(t,h) \subset P(t)$ $(U \div \mathcal{U}(p))$ $(\mathcal{V}(p) = \mathcal{V}(t,h) \subset Q(t)$ $(V \div \mathcal{V}(p)))$.

2) Strategy $U \div \mathcal{U}(p)$ $(V \div \mathcal{V}(p))$ is an admissible on the interval $[t_*, t^*]$, if $\mathcal{U}(t,h)$ $(\mathcal{V}(t,h))$, $t \in [t_*, t^*]$, are convex, closed and upper semi-continuous, that is: if $t_i \to t+0, t_i, t \in [t_*, t^*]$, $h_i \to h \in H$, then for any $\varepsilon > 0$ we have $\mathcal{U}(t_i, h_i) \subset \mathcal{U}^\varepsilon(t,h)$ $(\mathcal{V}(t_i,h_i) \subset \mathcal{V}^\varepsilon(t,h))$ for sufficiently large i.

3) $U_T \div \mathcal{U}(t,h) = P(t)$ $(V_T \div \mathcal{V}(t,h) = Q(t))$ are called the trivial strategies.

4) $U_u \div \mathcal{U}(t,h) = \{u(t)\}, \{u(\cdot)\}$ $(V_v \div \mathcal{V}(t,h) = \{v(t), v(t) = \{v(\cdot)\}\})$ are called program strategies.

Definition. A motion (on $[t_*, t^*]$) $x[t, h_*, U, V]$ from a position $p_* = \{t_*, h_*\}$, corresponding admissible (on $[t_*, t^*]$) strategies $U \div \mathcal{U}(p)$, $V \div \mathcal{V}(p)$, is a function $x[t] : [t_* - \tau, t^*] \to R^n$, piece-wise differentiable on $[t_*, t^*]$, satisfying the initial condition

$$x_{t_*} = h_*, \tag{3}$$

and for all $t \in [t_*, t^*]$ the equation

$$\begin{aligned}\dot{x}[t] = A(t)x[t] + A_\tau(t)x[t-\tau] &+ B(t)u[t] \\ &- C(t)v[t] + w(t),\end{aligned} \tag{4}$$

where $u[t]$, $v[t]$ are piece-wise continuous functions and for all $t \in [t_*, t^*] : u[t] \in \mathcal{U}(t, x_t), v[t] \in \mathcal{V}(t, x_t)$.

A motion (on $[t_*, t^*]$) $x[t, p_*, U_u, V_v]$ from a position $p_* = \{t_*, h_*\}$, corresponding program strategies $U_u \div \{u(t)\}$, $(V_v \div \{v(t)\})$, is a function $x[t] : [t_* - \tau, t^*]$ piece-wise differentiable on $[t_*, t^*]$, satisfying the condition (3) and for all $t \in [t_*, t^*]$ the equation

$$\begin{aligned}\dot{x}[t] = A(t)x[t] + A_\tau(t)x[t-\tau] &+ B(t)u(t) \\ &- C(t)v(t) + w(t).\end{aligned} \tag{5}$$

Theorem 1. For any position p_*, and admissible strategies U, V there exists the corresponding motion $x[t, p_*, U, V]$.

Existence of the motion $x[t, p_*, U_u, V_v]$ follows from the Cauchy formula.

We say that admissible strategy U of the first player guarantees approach of the system (1) with a set $M \subset H$ from a position $p_0 = \{t_0, p_0\}$ at a moment $\vartheta \ge t_0$, if for every motion $x[t] = x[t, p_*, U, V]$ $x_\vartheta \in M$.

An approach problem. A position $p_0 = \{t_0, p_0\}$, $\vartheta \ge t_0$, and convex closed set $M \subset H$ are given. The goal is to construct an admissible strategy U^0 of the first player which guarantees the approach of the system (1) with the set M from the position p_0 to the moment ϑ.

Obviously that without loss of generality one can consider that M is the bounded set: further, in this section, we investigate the approach problem assuming the boundedness of the set M.

Definition. A system of nonempty sets $W_t \subset H$, $t_0 \le t \le \vartheta$, is strong u-stable, if for any t_*, $t^* \in [t_0, \vartheta]$, $t_* \le t^*$, $h_* \in W_{t_*}$, $v \in \{v(\cdot)\}$ there exists $u \in \{u(\cdot)\}$ such that the motion $x[t] = x[t, \{t_*, h_*\}, U_u, V_v]$ satisfies the condition $x_{t^*} \in W_{t^*}$.

Denote:

- $dis(h, W_t) = \min_{g \in W_t} \|h - g\|_H$
- $Z(t,h) = \{h^* = \{x^*, y^*(\cdot)\} \in W_t | \|h^* - h\|_H = dis(h, W_t)\}$
- $Z_x = \{x^* | h^* \in Z(t,h)\}$
- $h = \{x, y(\cdot)\} \in H$

Definition. The strategy (of the first player) is *extremal* to a system of nonempty sets $W_t \subset H$, $t_0 \le t \le \vartheta$, is

$$U \div \mathcal{U}(t,h) = \{u^* | [z_{t,h} - x(0)]' B(t) u^* = \max_{u \in P(t)} [z_{t,h} - x(0)]' B(t) u\}, \quad t \in [t_0, \vartheta],$$

where $z_{t,h}$ is the nearest to x vector from $Z_x = \{x^* | \|h^* - h\|_\tau = min_{g \in W_t} \|g - h\|_\tau\}$.

4. SOLVABILITY CONDITIONS

Theorem 2. Let system of convex closed sets W_t, $t_0 \le t \le \vartheta$, is strong u-stable, $W_\vartheta \subset M$, and $h^0 \in W_{t_0}$. If extremal strategy U to a system of sets W_t is admissible on $[t_0, \vartheta]$, then it solves the *approach problem.*

Thus, in accordance with the methodology of i-smooth analysis, to solve the approach problem, we will aim at the finite-dimensional component, that is,

$$u^* = \max_{u \in P(t)} [z_{0,h_0} - x(t)]' B(t) u, \quad t \in [t_0, \vartheta].$$

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